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A note on the Kraichnan–Phillips theorem

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The Kraichnan–Phillips theorem asserts that, for incompressible, homogeneous, turbulent flow over a plane, rigid wall, the wall-pressure wavenumber–frequency spectrum $P(\boldsymbol{k}, \omega) \rightarrow 0$ as the planar wavenumber $\boldsymbol{k} \rightarrow 0$ provided the frequency $\omega \neq 0$. A proof of this theorem is given by use of a general formula that expresses the normal force on an arbitrary rigid body in terms of volume and surface integrals involving the vorticity. Implications for the theory of flow-induced surface vibrations are briefly discussed.

1. Introduction

Structural vibrations are produced by turbulent-boundary-layer wall pressures (cf. Chandiramani 1977, 1983; Sevik 1983). For a plane fluid–solid interface and homogeneous substructure, the excitation of vibrations is dominated by the low-wavenumber (*subconvective*) region of the wall-pressure wavenumber–frequency spectrum $P(\boldsymbol{k}, \omega)$, where the wavenumber $\boldsymbol{k}$ is conjugate to spatial position in the plane of the wall, and ω denotes frequency (Chase 1980, 1987). In flow at small Mach number the peak wall pressures occur in the *convective* domain centred on a streamwise wavenumber of order ω/U_c, where U_c is a convection velocity equal to about 70 % of the velocity of the main stream. A low-wavenumber component of wall pressure is typically 30–50 dB smaller, but can effect a considerable transfer of mean flow energy to the vibrating structure when its wavelength matches that of a resonant wall mode.

The wall-pressure spectrum has been subject to intense experimental and theoretical scrutiny in recent years (see Blake 1986; Chase 1980, 1987; Ffowcs Williams 1982; and references cited therein). Convective pressures correlate well with frequency ω, friction velocity v_* and boundary-layer thickness δ, but the precise levels in the low-wavenumber region ($k\delta \lesssim O(1)$) are elusive, and measurements vary widely from one experiment to another. Theoretical work by Phillips (1956) and Kraichnan (1956) suggest that $P(\boldsymbol{k}, \omega) \rightarrow 0$ as $\boldsymbol{k} \rightarrow \boldsymbol{0}$ for incompressible flow over a rigid wall, a result frequently referred to as the Kraichnan–Phillips theorem. Chase (1991) has argued that the theorem is strictly valid only in an inviscid approximation, and that $P(\boldsymbol{0}, \omega)$ can be non-zero when proper account is taken of the no-slip condition at the wall. That conclusion is important because it appears to accord with experimental studies which indicate that $P(\boldsymbol{k}, \omega)$ is wavenumber-white at subconvective wavenumbers (Leehey 1988), and also because (as Chase 1991 demonstrates) it implies that the associated wall shear stress spectrum ($\tau_{ij}(\boldsymbol{k}, \omega)$, say) behaves in a similar manner as $\boldsymbol{k} \rightarrow \boldsymbol{0}$. Non-zero values of $\tau_{ij}(\boldsymbol{k}, \omega)$ for small $\boldsymbol{k}$ could explain the excitation of long-wavelength, longitudinal waves in elastomer coated cylinders, observed in tests involving turbulent flow parallel to the cylinder axis.

However, the purpose of the present note is to supply alternative theoretical support for the Kraichnan–Phillips theorem, whose validity (it will be claimed) is maintained even under the circumstances contemplated by Chase. This conclusion does not invalidate Chase's (1991) formal analysis (which relates $P(\boldsymbol{k},\omega)$ and $\tau_{ij}(\boldsymbol{k},\omega)$ at low wavenumbers), but does conflict with his assertion that the spectra remain non-zero and wavenumber-white as $\boldsymbol{k}\rightarrow\boldsymbol{0}$.

The analysis is given in §2. A general formula given by Howe (1989) is used to determine the force exerted on a large, plane rigid plate by a compact region of vorticity in incompressible flow. This is generalized to turbulent boundary-layer flow, and it is deduced that $P(\boldsymbol{0},\omega)\rightarrow 0\ (\omega \neq 0)$ as the dimensions of the wall become infinite.

2. Analysis

2.1. *Normal force exerted on a large plate by a compact region of vorticity*

A compact distribution of vorticity in incompressible fluid of uniform density ρ and shear viscosity μ is adjacent to a rigid plate of infinite span and chord $2a$ in the absence of a mean flow (see figure 1*a*). The region where the vorticity is significantly different from zero has characteristic dimension l, and will be designated the 'eddy'. It is assumed that Reynolds number $vl/\nu \gg 1$, where v, $\nu\,(=\mu/\rho)$ respectively denote the characteristic velocity within the eddy and the kinematic viscosity. Viscous forces are therefore important only at exterior points of the eddy in the boundary layers induced on the plate.

Introduce a coordinate system $\boldsymbol{x} = (x_1, x_2, x_3)$ such that the plate occupies the region $|x_1| < a$, $x_2 = 0$, $-\infty < x_3 < \infty$, and assume the eddy to be located above the plate ($x_2 > 0$) in the vicinity of the origin O. Let $\boldsymbol{\Omega}_{\mathrm{e}}(\boldsymbol{x},t)$ denote the vorticity within the eddy. The total vorticity $\boldsymbol{\Omega}$ is the sum of $\boldsymbol{\Omega}_{\mathrm{e}}$ and that generated at the surfaces of the plate.

In incompressible flow at velocity $\boldsymbol{v}$, the normal force F (in the $-x_2$-direction) on the plate can be expressed in the form (Howe 1989)

$$F = -\rho\int \nabla X\cdot\boldsymbol{\Omega}\wedge\boldsymbol{v}\,\mathrm{d}^3\boldsymbol{x} - \mu\oint_S \nabla X\wedge\boldsymbol{\Omega}\cdot\mathrm{d}\boldsymbol{S}. \tag{2.1}$$

The volume integral is taken over the fluid and the surface integral is over the surfaces of the plate, the surface element $\mathrm{d}\boldsymbol{S}$ being directed into the fluid. $X(\boldsymbol{x})$ is an harmonic function equal to the velocity potential of irrotational flow past the plate that has zero circulation about the plate, and unit speed in the x_2-direction at large distances from the plate, i.e.

$$X = \mathrm{Re}\{-\mathrm{i}(z^2-a^2)^{\frac{1}{2}}\}, \quad z = x_1 + \mathrm{i}x_2. \tag{2.2}$$

Equation (2.1) will be used to determine the behaviour of F when the chord of the plate is very much larger than the eddy scale $l\,(a/l \gg 1)$.

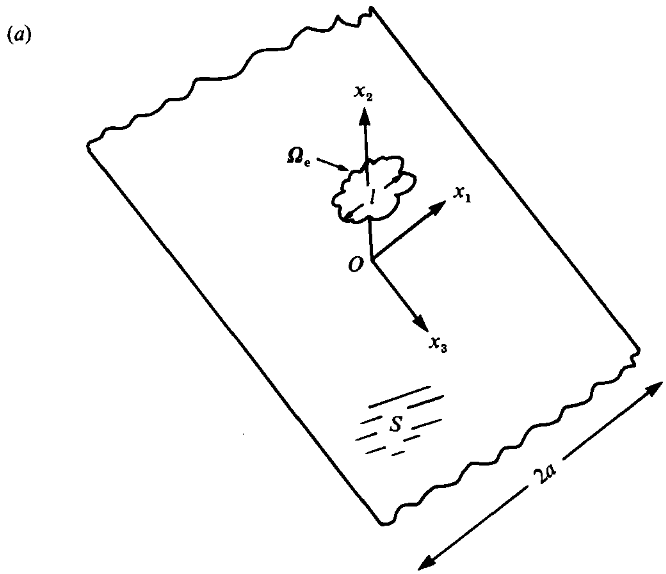
When $r \equiv |\boldsymbol{x}| \ll a$,

$$\nabla X \approx \boldsymbol{Z}/a, \quad \boldsymbol{Z} = (-x_1, x_2, 0). \tag{2.3}$$

Thus, the asymptotic behaviour of F is given by

$$F = -\frac{\rho}{a}\int \boldsymbol{Z}\cdot\boldsymbol{\Omega}\wedge\boldsymbol{v}\,\mathrm{d}^3\boldsymbol{x} - \frac{\mu}{a}\oint_S \boldsymbol{Z}\wedge\boldsymbol{\Omega}\cdot\mathrm{d}\boldsymbol{S}, \quad \frac{a}{l}\rightarrow\infty, \tag{2.4}$$

provided the integrands in (2.4) decay sufficiently fast with distance r from the eddy. To establish the validity of this formula we proceed to examine the behaviour of the integrands at positions on, and in the neighbourhood of, the plate for $l \ll r \ll a$.



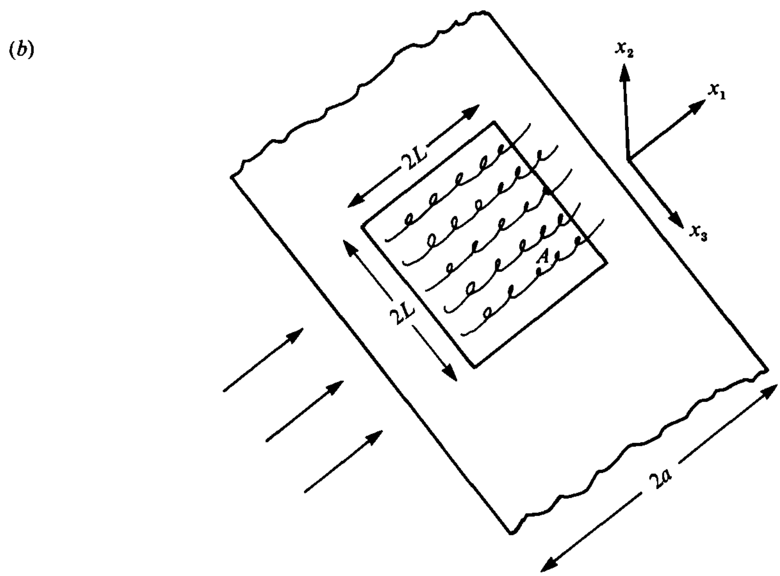


FIGURE 1. (*a*) Compact region of vorticity of lengthscale l adjacent to a rigid plate in a viscous, incompressible fluid. (*b*) Turbulent flow over the plate. The wavenumber–frequency spectrum is defined on the square region A of side $2L \ll a$.

When the plate is absent, the pressure $p(\boldsymbol{x}, t)$ can be calculated from the solution of the equation

$$\nabla^2\{p/\rho + \tfrac{1}{2}v^2\} = -\nabla\cdot(\boldsymbol{\Omega}_e \wedge \boldsymbol{v}), \tag{2.5}$$

which is obtained by taking the divergence of the momentum equation. Since $\boldsymbol{v}$ decays at least as fast as $1/r^3$ as $r/l \to \infty$, and

$$\int_{-\infty}^{\infty} (\boldsymbol{\Omega}_e \wedge \boldsymbol{v})(\boldsymbol{y}, t)\, \mathrm{d}^3\boldsymbol{y} \equiv \mathbf{0}$$

for a closed system of vorticity (Batchelor 1967), we find,

$$p \sim \rho v^2 (l/r)^3, \quad r \gg l. \tag{2.6}$$

The same order of magnitude estimate is applicable in the presence of the plate (because any surface-generated vorticity at large distances from the eddy is produced by the action of the pressure field), and may be used to determine the magnitude of the surface and boundary-layer vorticities when $l \ll r \ll a$. For fluctuations of frequency ω, the linearized momentum equation and (2.6) imply that the inviscid, irrotational velocity $\boldsymbol{v}_e$ accompanying the pressure fluctuations (2.6) satisfies,

$$v_e \sim \frac{v^2}{\omega l}(l/r)^4, \quad l \ll r \ll a. \tag{2.7}$$

The lengthscale of variation of $\boldsymbol{v}_e$ is equal to r when $l \ll r \ll a$. Thus, when account is taken of viscosity and the no-slip condition at the surface of the plate, the velocity distribution $\boldsymbol{u}$, say, within the unsteady boundary layer may be determined from the linearized boundary-layer equation (in the absence of mean flow)

$$\frac{\partial \boldsymbol{u}}{\partial t} + \nu \frac{\partial^2 \boldsymbol{u}}{\partial x_2^2} = \frac{\partial \boldsymbol{v}_e}{\partial t}, \quad l \ll r \ll a, \tag{2.8}$$

where $\boldsymbol{u} \rightarrow \boldsymbol{v}_e$ outside the boundary layer. We easily calculate from this (Lighthill 1978, Chap. 2) that the surface distribution of vorticity is given in order of magnitude by

$$\boldsymbol{\Omega} = \frac{\partial \boldsymbol{u}}{\partial x_2} \sim \frac{v}{l}\left(\frac{v^2}{\omega \nu}\right)^{\frac{1}{2}}\left(\frac{l}{r}\right)^4, \quad x_2 = 0, \quad l \ll r \ll a. \tag{2.9}$$

It follows from (2.7), (2.9) that the components of frequency ω of the integrands on the right of (2.4) are at most of order $1/r^6$ and $1/r^3$ respectively when $l \ll r \ll a$. These estimates confirm the absolute convergence of both integrals, and that the order of magnitude of the force can be cast in the form

$$F \sim \frac{l}{a}\rho v^2 l^2 f, \tag{2.10}$$

where f is a dimensionless function of ω that does not depend on a. This shows that the interaction of the eddy with the plate produces a non-vanishing normal force only by virtue of edge effects, i.e. as a consequence of the edges of the plate being within the hydrodynamic near field of the eddy.

Let us consider how the argument must be modified by the presence of a laminar or turbulent mean flow over the plate in the positive x_1-direction. We are concerned solely with that component F of the normal force that can be attributed to the eddy (i.e. the component that is coherent with the eddy vorticity). Additional inertia terms of the form $\boldsymbol{v}\cdot\nabla\boldsymbol{v}$ must now be included in the boundary-layer equation (2.8). Their principal effect is to modify the structure of the boundary layer associated with the component $\boldsymbol{u}$ of the net velocity $\boldsymbol{v}$ which is equal asymptotically to $\boldsymbol{v}_e$ outside the boundary layer. However, the new terms can have no influence on the order of magnitude of $\boldsymbol{u}$, which is still determined by the 'incident' pressure gradient (produced by the eddy) that drives the unsteady velocity $\boldsymbol{v}_e$. This implies the continued convergence of the surface integral in (2.4) when $\boldsymbol{\Omega}$ is identified with the component of surface vorticity coherent with the eddy.

For the volume integral of (2.4), the coherent part of the integrand in the range $l \ll r \ll a$ includes terms linear in the mean boundary-layer vorticity and velocity.

Elementary calculation reveals that the contribution to the volume integral from the boundary layers, per unit surface area of the plate, is of order $Uv_e\,\delta_*$, where U is the mean flow velocity and δ_* is the displacement thickness of the mean boundary layer. Convergence of the integral is ensured by noting that δ_* certainly grows no faster than x_1 (Hinze 1975), and v_e decreases as $1/r^4$ when $r \gg l$.

We conclude that the estimate (2.10) for the normal force produced by the eddy remains valid in the presence of a mean boundary-layer flow.

2.2. *The Kraichnan–Phillips theorem*

Consider next turbulent boundary-layer flow over the upper surface of the plate, whose chord $2a$ is taken to be very large. We examine the wall-pressure fluctuations in the square region A, $|x_1| < L$, $|x_3| < L$ of side $2L$, where $L \ll a$ (see figure 1*b*). The turbulence in A is assumed to be homogeneous in planes parallel to the wall, with correlation scale $l \approx \delta \ll L$, where the variation of the boundary-layer thickness δ within A is small. These conditions are implicitly assumed in the usual definition of the wall-pressure spectrum $P(\boldsymbol{k},\omega)$.

Let $p(x_1,x_3,t)$ denote the wall pressure in A, and define Fourier coefficients $\alpha_{mn}(\omega)$ by

$$\alpha_{mn}(\omega) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-L}^{L}\int_{-L}^{L} p(x_1,x_3,t)\exp\{-\mathrm{i}(m\pi x_1/L+n\pi x_3/L-\omega t)\}\,\mathrm{d}x_1\,\mathrm{d}x_3\,\mathrm{d}t. \tag{2.11}$$

For time-stationary turbulence that is homogeneous in planes parallel to the plate, the wall-pressure spectrum is given by

$$P(k_1,k_3,\omega) = \lim_{L/\delta\to\infty}\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-L}^{L}\int_{-L}^{L} R(x_1,x_3,t)\exp\{-\mathrm{i}(k_1x_1+k_3x_3-\omega t)\}\,\mathrm{d}x_1\,\mathrm{d}x_3\,\mathrm{d}t, \tag{2.12}$$

where $R(x_1,x_3,t) = \langle p(x_1',x_3',\tau)\,p(x_1+x_1',x_3+x_3',t+\tau)\rangle$ is the wall-pressure correlation function, and angle brackets denote an ensemble average. It follows from this and (2.11) that

$$\langle \alpha_{mn}(\omega)\,\alpha_{MN}^*(\omega')\rangle = \frac{L^2}{\pi^2}P\left(\frac{m\pi}{L},\frac{n\pi}{L},\omega\right)\delta_{Mm}\,\delta_{Nn}\,\delta(\omega-\omega'),\quad L \gg \delta, \tag{2.13}$$

where the asterisk indicates complex conjugate.

According to (2.11),

$$\alpha_{00}(\omega) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-L}^{L}\int_{-L}^{L} p(x_1,x_3,t)\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}x_1\,\mathrm{d}x_3\,\mathrm{d}t. \tag{2.14}$$

If $\delta \ll L \ll a$, the order of magnitude of this integral can be estimated from (2.10) to be

$$\alpha_{00}(\omega) \sim \frac{l}{a}\rho v^2 l^2 \sum_n \hat{f}_n(\omega), \tag{2.15}$$

where the summation is over the assembly of statistically independent boundary-layer eddies in A (or within a region enclosing A whose linear dimensions are larger by a finite multiple of the turbulence correlation length $l \sim \delta$), and $\hat{f}_n(\omega)$ is the Fourier time transform of a stationary random function of time, and corresponds to the factor f in (2.10) for the nth eddy.

Since the flow in A is homogeneous, there exists a common frequency spectrum $\Phi(\omega)$ such that

$$\langle \hat{f}_n(\omega)\hat{f}_n^*(\omega')\rangle = \Phi(\omega)\,\delta(\omega-\omega'). \tag{2.16}$$

When $L \gg l$ there are approximately $(2L/l)^2$ statistically independent eddies in A. It follows that $\langle \sum_m \sum_n \hat{f}_m(\omega)\hat{f}_n^*(\omega')\rangle \approx (2L/l)^2\Phi(\omega)\,\delta(\omega-\omega')$, and therefore that (2.13)–(2.15) imply

$$P(0,0,\omega) \sim \left(\frac{l}{a}\right)^2 l^2(\rho v^2)^2\Phi(\omega) \sim \left(\frac{\delta}{a}\right)^2 \delta^2(\rho v^2)^2\Phi(\omega). \tag{2.17}$$

The Kraichnan–Phillips theorem follows on taking the limit $a/l \rightarrow \infty$.

This result says nothing about the behaviour of $P(\boldsymbol{k},\omega)$ for $\boldsymbol{k} \neq \boldsymbol{0}$. In a formal expansion $P(\boldsymbol{k},\omega) = P(\boldsymbol{0},\omega) + k_j\partial P(\boldsymbol{0},\omega)/\partial k_j + \ldots$, the coefficient of each term in the series is influenced by the finite size of the plate, and presumably tends to a unique limit (consistent with the local properties of the turbulent boundary layer) as $\delta/a \rightarrow 0$. In that limit the variation of $P(\boldsymbol{k},\omega)$ for small but finite $\boldsymbol{k}$ must scale on the boundary-layer thickness δ, and the first non-trivial term in the expansion is usually assumed to be $O(k^2\delta^2)$ (cf. Chase 1987).

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